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1985 J. Phys. A: Math. Gen. 18 L359

(http://iopscience.iop.org/0305-4470/18/7/005)

J. Phys. A: Math. Gen. 18 (1985) L359-L361. Printed in Great Britain

LETTER TO THE EDITOR

Double Laplace transform of the Coulomb Green function

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Received 30 October 1984

Abstract. A differential equation method is used to derive a closed form expression for the double Laplace transform of the l wave reduced Coulomb Green function in terms of Gaussian hypergeometric functions. Our expression is found to reproduce in a rather natural way the special results by earlier workers.

The double and iterated Laplace transforms of the partially projected reduced Coulomb Green function

$$G_l(r, r'; E) = (rr')^{l+1} g_l(r, r'; E) \tag{1}$$

for an arbitrary angular momentum l and energy E play a role in the theories of scattering by Coulomb plus short-range separable potentials with exponential form factors. For example, they determine the movement of the bound state pole of the Hamiltonian

$$H = -\tfrac{1}{2}\nabla^2 - Zr^{-1} + \sum_{i=1}^{N} \lambda_i \exp[-\alpha_i(r+r')] \tag{2}$$

as a function of the coupling constant λ_i of the separable interaction (Kok and van Haeringen 1981). These transforms are also useful for the computation of lower bounds to the eigenvalues of the atomic Hamiltonian (Hill and Huxtable 1982) with atomic number Z. In the traditional approach, one contemplates expressing the double Laplace transform

$$\begin{aligned}\bar{G}_l(\beta, \beta'; E) &= L_2[G_l(r, r'; E); \beta, \beta'] \\ &= \int_0^\infty \mathrm{d}r \int_0^\infty \mathrm{d}r' \exp(-\beta r - \beta' r') G_l(r, r'; E)\end{aligned} \tag{3}$$

in closed form by using the known expression for l wave off-shell Coulomb t matrix t_l^c. Unfortunately, the expression for t_l^c in momentum space is extremely complicated (Dušek 1981, van Haeringen 1983); thus the evaluation of the general integral in equation (1) has been restricted to a few lower partial waves (van Haeringen and van Wageningen 1975, van Haeringen 1977) only. In this letter we derive a differential equation method to write a closed form expression for $\bar{G}_l(\beta, \beta'; E)$ without any constraint on l and E.

The Green function $g_l(r, r'; E)$ satisfies the radial equation

$$(H_l - E) g_l(r, r'; E) = -(r, r')^{-1} \delta(r - r') \tag{4}$$

0305-4470/85/070359+03$02.25

with

$$H_l = -\frac{1}{2r}\frac{\partial^2}{\partial r^2}r + \frac{l(l+1)}{2r^2} - \frac{Z}{r}. \tag{5}$$

From equations (1), (4) and (5) we have

$$\frac{\partial^2 G_l(r,r';E)}{\partial r^2} - \frac{2l}{r}\frac{\partial G_l(r,r';E)}{\partial r} + \frac{2Z}{r}G_l(r,r';E) + 2EG_l(r,r';E)$$
$$= 2(rr')^l \delta(r-r'). \tag{6}$$

Taking the double Laplace transform of equation (6) with respect to β, β' we obtain

$$\left(\beta^2 - \frac{Z^2}{\nu^2}\right)\frac{\partial \tilde{G}_l(\beta,\beta';E)}{\partial \beta} + [2(l+1)\beta - 2Z]\tilde{G}_l(\beta,\beta';E) = -\frac{2}{(\beta+\beta')^{2l+2}} \tag{7}$$

with

$$\tilde{G}_l(\beta,\beta';E) = \bar{G}_l(\beta,\beta';E)/(2l+1)!. \tag{8}$$

In writing equation (7) we have introduced

$$\nu = Z(-2E)^{-1/2}. \tag{9}$$

We now look for a solution of equation (7) in the form

$$\tilde{G}_l(\beta,\beta';E) = \frac{(1-\eta)}{(2\nu Z)^{2l+2}(\beta+\beta')^{2l+2}}f(\eta) \tag{10}$$

with

$$\eta = (\nu\beta' - Z)(\nu\beta - Z)/(\nu\beta' + Z)(\nu\beta + Z). \tag{11}$$

Combining equations (7), (10) and (11) and differentiating the resulting equation with respect to η we get

$$\eta(1-\eta)f''(\eta) + [(l-\nu+2) + (l+\nu-2)\eta]f'(\eta) + (l+\nu)f(\eta) = 0. \tag{12}$$

The solution of equation (12) is given by

$$f(\eta) = A\,{}_2F_1(1, -l-\nu; l-\nu+2; \eta) \tag{13}$$

where A is a constant to be determined from the boundary conditions on $\tilde{G}_l(\beta,\beta';E)$. From equations (10), (11) and (13) we write

$$\tilde{G}_l(\beta,\beta';E) = \frac{A}{(\nu\beta'+Z)(\nu\beta+Z)(2\nu Z)^{2l+1}(\beta+\beta')^{2l+1}}$$
$$\times {}_2F_1(1, -l-\nu; l-\nu+2; (\nu\beta'-Z)(\nu\beta-Z)/(\nu\beta'+Z)(\nu\beta+Z)). \tag{14}$$

By invoking the requirements of the Watson's lemma (Watson 1981), which determines the behaviour of the Laplace transform in equation (14) for large values of β and β' from a small r (small r') expansion (Newton 1966) of the radial Green function, we obtain A in the form

$$A = \nu^2(2\nu Z)^{2l+1}/(l-\nu+1). \tag{15}$$

Combining equations (8), (14) and (15) we write our final result for $\bar{G}_l(\beta,\beta';E)$ as

$$\bar{G}_l(\beta,\beta';E) = \frac{(2l+1)!\,\nu^2}{(l-\nu+1)(\nu\beta'+Z)(\nu\beta+Z)(\beta+\beta')^{2l+1}}$$
$$\times {}_2F_1(1, -l-\nu; l-\nu+2; (\nu\beta'-Z)(\nu\beta-Z)/(\nu\beta'+Z)(\nu\beta+Z)). \tag{16}$$

The iterated Laplace transform of $G_l(r, r'; E)$ is a special instance of equation (16) when $\beta = \beta'$. We conclude by noting that for $l = 0, 1, 2, \ldots$, etc, our general expression in equation (16) yields the corresponding results obtained by earlier workers in different publications (van Haeringen 1981).

This work is part of the project 'some studies in few body problems' supported by the Department of Atomic Energy Government of India. One of the authors (SRB) is grateful to the University Grants Commission for a research grant.

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